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LETTER TO THE EDITOR

Conservative dynamics, phase ordering and stretched exponential decays

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Abstract. We discuss the dynamics of an observable field related to a locally conserved field. We show that for a model of phase ordering dynamics and for a variation of model C critical dynamics, that the two-time momentum space correlation function decays asymptotically as a stretched exponential. We show that under some conditions the asymptotic behaviour is experimentally observable and argue that the non-exponential behaviour is not a feature specific to these models but should rather apply for any observable coupled to a locally conserved field.

It is often found that the asymptotic decay of temporal correlation functions are of the stretched exponential form [1]. One way in which a stretched exponential decay arises is if there is a broad distribution of relaxation times τ such that the distribution vanishes sufficiently slowly as $\tau \to \infty$. Such a distribution may result from dynamics on a hierarchical structure [2] or be due to a distribution of lengths as in polymers solutions [3]. In these cases the distribution of relaxation times is due to some specific property of each individual system. On the other hand, a broad distribution of relaxation times occurs naturally if the dynamics obey a local conservation law. In this letter we will demonstrate that the presence of a local conservation law can lead to a stretched exponential form a two-time correlation functions.

The effect of conservation laws is of crucial importance in understanding the dynamics of any system. For example, it has recently been shown that, due to the distribution of relaxation times, conservative dynamics generically leads to an asymptotic *power law decay* for the two-point real-space correlation function [4]. In this letter we will consider the dynamics of an observable coupled to a locally conserved field. We study two models in which this situation is realized and show that the presence of the conserved field leads to a stretched exponential form for the asymptotic decay of the two-time momentum space correlation function. The first is a model of phase ordering dynamics introduced by Ohta *et al* [5], and the second model can be considered to be a variation of model C critical dynamics as defined by Hohenberg and Halperin [6]. We discuss the circumstances under which the asymptotic decay is experimentally observable and argue that the non-exponential decay is not a special feature of these models but, rather, is a general feature of an observable, nonlinearly coupled to a conserved field.

We first consider the model of phase ordering dynamics without conservation of order parameter introduced by Ohta, Jasnow and Kawasaki (OJK) [5]. The phase ordering process occurs after a rapid quench of a binary alloy or fluid into its miscibility gap. Initial attention was focused on the single time dynamical scaling behaviour [7]. More recently there has been increasing interest in two-time correlations. This interest arises firstly to obtain information about the instantaneous behaviour [8], secondly to further differentiate dynamical universality classes [9, 10], and thirdly to determine other universal features of phase ordering dynamics [11]. These studies have typically been restricted to correlations in real space [12], to macroscopic densities such as the global magnetization [11] or to systems with vector order parameters [9]. On the other hand, the experimental measurements have been of the correlation function at a finite wavevector k for systems with single-component order parameters [13]. There has also been simulations of the kinetic Ising model [14].

The presence of sharp interfaces makes the late stage phase separation process analytically difficult. Ohta *et al* circumvented this problem by introducing a spatially smooth auxiliary field $u(\boldsymbol{r}, t)$ where $\boldsymbol{r}$ is the position and t is the time after the quench. The observable order parameter field $\psi(\boldsymbol{r}, t)$ is related to $u(\boldsymbol{r}, t)$ by a nonlinear mapping $\psi(\boldsymbol{r}, t) = \psi_{\text{eq}} \operatorname{Sign}(u(\boldsymbol{r}, t))$ where ψ_{eq} is the magnitude of the equilibrium value of ψ. The dynamics of the u-field is then obtained from the Cahn–Allen equations of motion [15] for the interfaces and by assuming that the u-field obeys the same equation of motion as that of its ensemble average. The OJK model is therefore a mean field approximation of the dynamics. With these assumptions it is found that the u field obeys a diffusion equation

$$\frac{\partial u(\boldsymbol{r}, t)}{\partial t} = D\nabla^2 u(\boldsymbol{r}, t) \tag{1}$$

where D is a constant related to the surface tension. Note that although the observable ψ field is not locally conserved by the dynamics (the time evolution does not obey the continuity equation), the u field is locally conserved. Therefore, in this example, the observable field is nonlinearly related to a locally conserved field.

A nice feature of the OJK model is that if u is a Gaussian random variable at $t = 0$ then u is a Gaussian random variable for all time $t > 0$. Therefore any expectation value can, in principle, be calculated. The scaling behaviour of the OJK model was found to agree well with experimental and simulation data. The OJK model exhibits dynamical scaling with the characteristic domain size growing as $t^{1/2}$ [5]. The quasistatic scattering function $\langle \hat{\psi}_{\boldsymbol{k}}(t)\hat{\psi}_{-\boldsymbol{k}}(t)\rangle$ [5, 17] the two-point correlation function $\langle \psi(\boldsymbol{r}_1, t)$-$(\boldsymbol{r}_2, t)\rangle$, and the two-time correlation function $\langle \psi(\boldsymbol{r}, t_1)\psi(\boldsymbol{r}, t_2)\rangle$ [16] were all found to be in excellent agreement with simulation results. To relate OJK to other models, note that the scaling behaviour of the OJK model is the same as that of the resummation method of Kawasaki, Yalabik and Gunton [18] and almost the same as that of a more sophisticated method of Mazenko [19]. The introduction of a spatially smooth auxiliary field and the mean field approximation are also fundamental steps in these methods. The differences arises from the step at which the Gaussian assumption is made.

For the OJK model the two-time, two-point order parameter correlation function is [16]

$$\langle \psi_1 \psi_2 \rangle = \psi_{\text{eq}}^2 \arcsin\left(\frac{\langle u_1 u_2 \rangle}{\langle u_1^2 \rangle^{1/2} \langle u_2^2 \rangle^{1/2}}\right). \tag{2}$$

Here $\psi_i = \psi(\boldsymbol{r}_i, t_i)$, $u_i = u(\boldsymbol{r}_i, t_i)$ and $\langle u_1 u_2 \rangle$ is the Green function for the diffusion equation so that

$$\begin{aligned}\frac{\langle u_1 u_2 \rangle}{\langle u_1^2 \rangle^{1/2} \langle u_2^2 \rangle^{1/2}} &\equiv \alpha(|\boldsymbol{r}_1 - \boldsymbol{r}_2|, t_1, t_2) \\ &= \left(\frac{2 l_1 l_2}{l_1^2 + l_2^2}\right)^{d/2} \exp\left(-\frac{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}{l_1^2 + l_2^2}\right)\end{aligned} \tag{3}$$

where we have set $D = 1/4$ for simplicity, and where $l_i = t_i^{1/2}$ is the time-dependent domain size and d is the spatial dimension.

The k-space correlation function is now easily obtained from (2) and (3)

$$
\begin{aligned}
S_k(t_1, t_2) &= \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \langle \psi_1 \psi_2 \rangle \\
&= \psi_{\mathrm{eq}}^2 \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} \left(\frac{2l_1 l_2}{l_1^2 + l_2^2}\right)^{dm+d/2} \\
&\quad \times \int \mathrm{d}\boldsymbol{r}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \exp\left(-\frac{(2m+1)r^2}{l_1^2 + l_2^2}\right) \\
&= \psi_{\mathrm{eq}}^2 (2\pi l_1 l_2)^{d/2} \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)} \left(\frac{1}{2m+1}\right)^{d/2} \\
&\quad \times \left(\frac{2l_1 l_2}{l_1^2 + l_2^2}\right)^{dm} \exp\left(\frac{-k^2(l_1^2 + l_2^2)}{2(2m+1)}\right)
\end{aligned}
\tag{4}
$$

where $\arcsin(\alpha)$ has been expanded in powers of α. The sum is absolutely convergent. For sufficiently large $l_2^2 = t_2$, the higher order terms in the sum will always dominate the lower order terms. Therefore the asymptotic decay is slower than any exponential.

For large argument $k^2(l_1^2 + l_2^2)$ the sum can be replaced by an integral

$$
\langle \hat{\psi}_k(t)_1 \hat{\psi}_{-k}(t_2) \rangle \sim (2\pi l_1 l_2)^{d/2} \int dm\, m^{-\delta} \left(\frac{2l_1 l_2}{l_1^2 + l_2^2}\right)^{dm} \exp\left(\frac{-k^2(l_1^2 + l_2^2)}{2(2m+1)}\right). \tag{5}
$$

We have assumed that the coefficients in the summation decreases as $m^{-\delta}$ for large δ. For the sum given by (4), $\delta = 3/2 + d/2$. For large argument the integral can be approximated as the maximum in the integrand. The integrand is maximized at $m = m^*$ given by

$$
2m^* + 1 = |\boldsymbol{k}| \left(d \frac{l_1^2 + l_2^2}{2 l_1^2 l_2^2}\right)^{1/2}. \tag{6}
$$

Therefore, for large t_2/t_1 the k space correlation function becomes

$$
\langle \hat{\psi}_k(t)_1 \hat{\psi}_{-k}(t_2) \rangle \sim \exp\left(-d \ln\left(\frac{l_1^2 + l_2^2}{2 l_1 l_2}\right) |\boldsymbol{k}| (l_1^2 + l_2^2)\right)^{1/2}. \tag{7}
$$

For fixed t_1 the asymptotic decay is, to within logarithmic corrections, a stretched exponential $\exp(-(t_2/t_1)^{1/2})$ for large t_2/t_1. The form of the asymptotic decay is independent of δ. This indicates that the result is independent of the form of the nonlinear map $\psi = f(u)$, except for rather general conditions. For example a polynominal mapping will lead to exponential decay and must be excluded.

Although the asymptotic decay is non-exponential, this does not mean that it is experimentally observable. An experiment, whether real or numerical, can only hope to see four or five decades in the decay of the correlation function. If $k^2 l_1^2 < 1$, the leading term in the sum (4) will dominate the first decades of the decay and the observed decay will seem exponential. However, if $k^2 l_1^2 > 1$, the contributions of the higher-order terms will be observable.

To test our arguments we evaluated the sum in (4) numerically. The first 500 terms in the sum are taken. The error can be shown to be negligibly small for the times considered. Since this is not a steady state we calculate the normalized correlation function $C_k(t_2/t_1) = S_k(t_1, t_2)/(S_k(t_1, t_1)S_k(t_2, t_2))^{1/2}$. Figure 1 shows the normalized correlation function plotted on a semi-log scale against t_2/t_1 for $d = 2$ and $kl_1 = 1/4$ (figure 1(*a*)) and $kl_1 = 1$ (figure 1(*b*)). We have arbitrarily defined the first four decades in the decay as experimentally observable. For $kl_1 = 1/4$ the decay is approximately exponential in this range, but the decay is slower than exponential for $kl_1 = 1$. Figure 2 shows the same normalized correlation against $(t_2/t_1)^{1/2}$. For $kl_1 = 1/4$ (figure 2(*a*)) the decay is faster than this stretched exponential form. For $kl_1 = 1$ one sees that the decay is approximately of the form $\exp((t_2/t_1)^{1/2})$. To demonstrate that this decay is asymptotic we show the same correlation function over 30 decades in figure 2(*c*). The very slight curvature in the plot is assumed to be due to the logarithmic corrections.

A comment should be made about comparing experimental data with the predictions of the OJK model. The smaller time t_1 in the correlation function must be within the scaling regime, i.e. the system must have well defined interfaces at t_1. This is because the OJK model is a description of the motion of the interfaces. In their study of spin glasses, Fisher and Huse [12] were interested in the correlation with the non-scaling initial state. This quantity cannot be calculated within the the OJK model which assumes well established domains at all times. This picture was confirmed in simulations in which it was found that the agreement with OJK improves as t_1 increased for fixed t_2/t_1 [16].

The k space intensity-intensity correlation function has been measured by Kim *et al* [13] for a binary fluid. However, one is more typically concerned with the two-time correlation functions in the steady state (or indeed in equilibrium). Therefore the correlation function for phase ordering dynamics is an unusual measurement. We note, however, that our primary result of a non-exponential decay of the correlation function is not limited to non-steady state dynamics. The non-exponential behaviour is due to the spectrum of relaxation times with the largest relaxation time being limited only by system size. The nonlinear relation between the observed ψ field and the hidden conserved field then couples in many relaxation times and gives the non-exponential decay for the wavevector modes of ψ. These features can also be realized in the steady state or even in equilibrium. Although we have emphasized the local conservation as

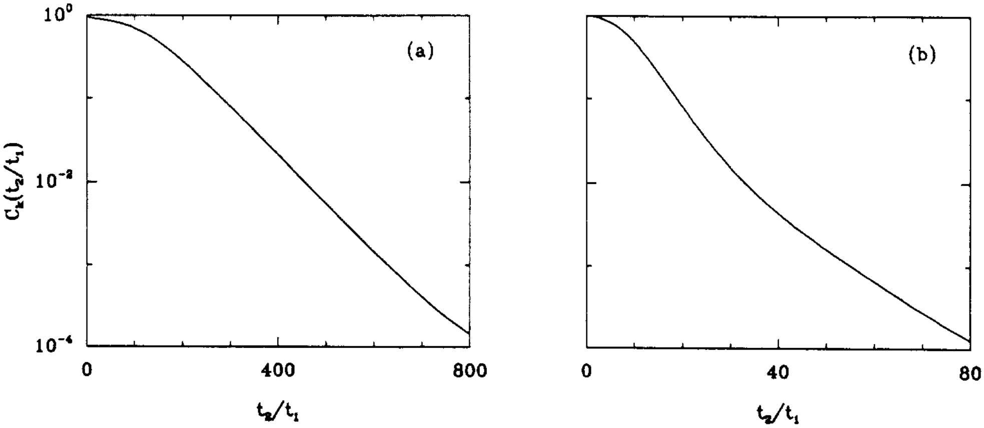
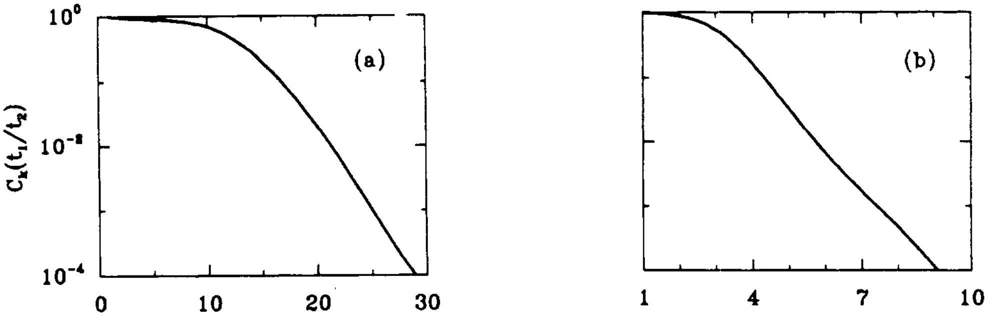


Figure 1. Semi-log plot of the normalized OJK correlation function against t_2/t_1 in the 'experimentally observable' regime. $kl_1 = 1/4$ in figure 1(*a*) and $kl_1 = 1$ in the figure 1(*b*). Within this range the decay for $kl_1 = 1/4$ appears approximately exponential, while for $kl_1 = 1$ there is a clear deviation from pure exponential decay.



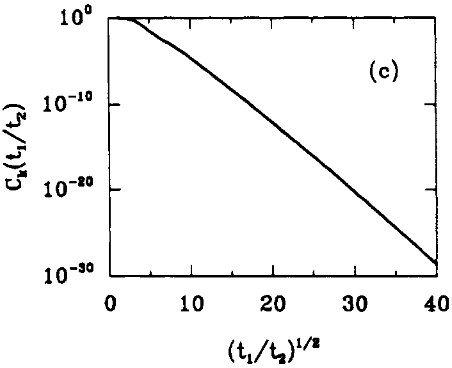


Figure 2. Semi-log plot of the normalized OJK correlation function against $(t_2/t_1)^{1/2}$. $kl_1 = 1/4$ in figure 1(a) and $kl_1 = 1$ in figure 1(b). Within this range the decay for $kl_1 = 1$ is approximately a stretched exponential. Figure 1(c) is the same as figure 1(b) except the plot is shown over 30 decades in the decay.

a natural way to obtain the distribution of relaxation times, coupling to any field with these characteristics will result in a stretched exponential decay. For example in the Oono–Puri extension of the OJK model [17], the u field is not conserved. However this field has the same distribution of relaxation times as the original OJK model and thus leads to the same asymptotic scaling behaviour.

To illustrate the result for a steady state we consider a variation of 'model C' dynamics of Hohenberg and Halperin [6]. In this model two fields are coupled as

$$\frac{\partial u(\boldsymbol{r}, t)}{\partial t} = \nabla^2(u + gu^3 - r_u^2\nabla^2 u + \varepsilon\psi) + \xi_u \tag{8}$$

$$\frac{1}{D}\frac{\partial\psi(\boldsymbol{r}, t)}{\partial t} = \psi - \psi^3 + r_\psi^2\nabla^2\psi + \varepsilon u + \xi_\psi \tag{9}$$

where ξ_u and ξ_ψ are Gaussian noise terms with correlation $\langle\xi_u(\boldsymbol{r}, t)\xi_u(\boldsymbol{r}', t')\rangle = -2T\nabla^2\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}')$. For example, in the application to crystallization, u is identified as the temperature and ψ is the order parameter. The operating parameters are chosen so that u is in the one phase region but ψ is in the two phase region. The u-field is locally conserved by the dynamics while the observable ψ-field is not.

The steady-state correlation function can be calculated in the limit $r_\psi = 0$, $\varepsilon \to 0^+$ and $1/D \to 0$. In this limit the dynamics of u are independent of ψ while the ψ-field is given by a nonlinear map of the conserved u field, i.e. $\psi(\boldsymbol{r}, t) = f(u(\boldsymbol{r}, t)) = \mathrm{sgn}(u(\boldsymbol{r}, t))$. If one further assumes that the nonlinearities of the u-field can be neglected, so that the u-field is a Gaussian random variable, the steady state ψ correlation function then follows directly from our previous result for the OJK model, namely

$$\langle\psi(\boldsymbol{r}, t)\psi(\mathbf{0}, 0)\rangle = \arcsin(\gamma(\boldsymbol{r}, t)) \tag{10}$$

where $\gamma(\boldsymbol{r}, t) \equiv \langle u(\boldsymbol{r}, t)u(\boldsymbol{0}, 0)\rangle / \langle u(\boldsymbol{r}, t)^2\rangle$ and $\langle u(\boldsymbol{r}, t)u(\boldsymbol{0}, 0)\rangle$ is the steady-state correlation function of u. The argument for the OJK model remains valid and the asymptotic decay of the k mode correlation function is a stretched exponential with exponent 1/2. The role of the initial correlation length l_{I} in the OJK model is replaced by the steady-state correlation length r_u of the u field. By analogy with the OJK model, for $kr_u > 1$ the non-exponential behaviour can be observed while for $kr_u < 1$ the experimentally observed decay will be approximately exponential.

To test these arguments we have explicitly calculated the correlation function for (8) with g and $\varepsilon = 0$ on a one-dimensional lattice. The lattice is 512 sites long with $r_u = 16$. Figure 3 shows a semi-log plot of the $\hat{\psi}_k$ correlation function against $\tau = t_2 - t_1$ for $kr_u = \pi/4$ (figure 3(a)), and $kr_u = \pi$ (figure 3(b)). As before the decay for the smaller, kr_u is approximately exponential within this range while for $kr_u = \pi$ the decay is non-exponential. Figure 4 shows the same correlation functions plotted against $\tau^{1/2}$. The decay is approximately a stretched exponential for $kr_u = \pi$.

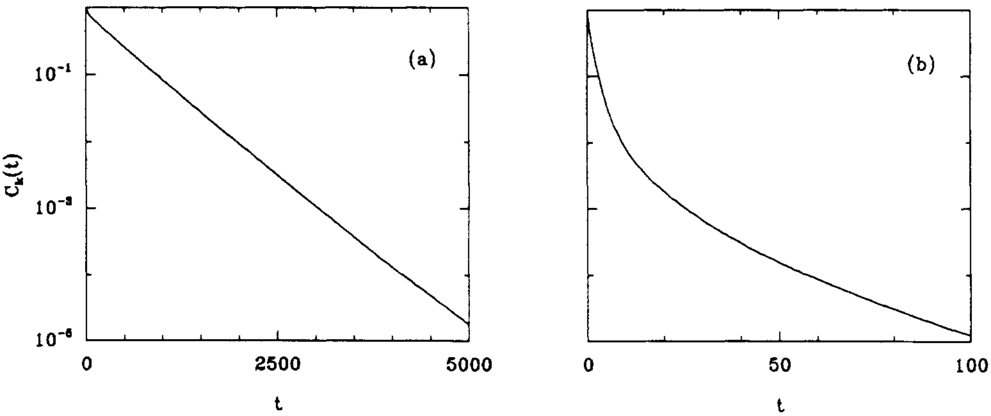


Figure 3. Semi-log plot of the correlation function for 'model C' plotted against $t = t_2 - t_1$. $kr_u = \pi/4$ in figure 3(a) and $kr_u = \pi$ in figure 3(b). The decay appears approximately exponential for $kr_u = \pi/4$.

The model we have considered here is somewhat unusual. In particular, the limits we have chosen are not the ones usually studied. However, this limit was chosen only for calculational simplicity; the fact that the asymptotic decay of the wavevector mode is non-exponential should be robust to changes in the model. Let us consider adding nonlinearities to dynamics of u in (8). In this case the u-field is no longer a Gaussian random variable so that the specific details of our calculation no longer holds. However there remains a spectrum of relaxation times due to the conservative dynamics. The longest relaxation time is only limited by the system size and nonlinear coupling of the observable ψ-field should still give a non-exponential decay in the correlation function. The non-exponential nature of the asymptotic decay should also not depend greatly on the way the ψ-field is coupled to the u-field. For example, allowing D to be finite or letting r_ψ be non-zero couples the dynamics of the u field to the observable ψ field. However, as noted in our analysis of the OJK model, the asymptotic non-exponential form of the decay is independent of the exact form of the mapping of the observable field to the underlying conserved field. Furthermore, the addition of coupling between the fields will likely result in a non-polynomial relation between the two fields.

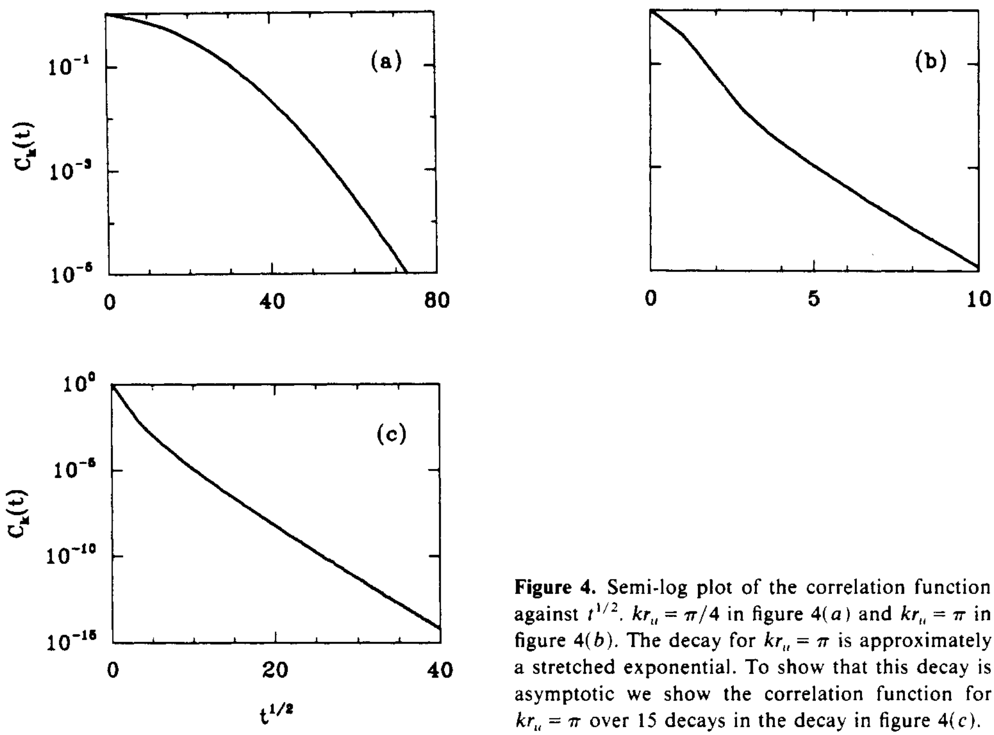


Figure 4. Semi-log plot of the correlation function against $t^{1/2}$. $kr_u = \pi/4$ in figure 4(*a*) and $kr_u = \pi$ in figure 4(*b*). The decay for $kr_u = \pi$ is approximately a stretched exponential. To show that this decay is asymptotic we show the correlation function for $kr_u = \pi$ over 15 decays in the decay in figure 4(*c*).

However, the stretch exponential exponent β of the stretch exponential, i.e. $c(t) \sim \exp(-t^{\beta})$ and especially, the observability of the non-exponential decay, may be affected.

To summarize, we have demonstrated that the asymptotic decay of the wavevector correlation function of two specific models is a stretched exponential with exponent 1/2. In both models the observable field is nonlinearly coupled to an underlying dynamically conserved field. The conservative dynamics naturally generate a spectrum of relaxation times, and the nonlinear relation between the fields results in a stretched exponential form for the asymptotic decay of the two-time correlation function in momentum space. We have demonstrated that for some circumstances the non-exponential behaviour is experimentally observable. Finally, we have argued that the non-exponential decay is not specific to these models but is a general feature of an observable nonlinearly related to a dynamically conserved field.

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